An effective singular oscillator for Duffin-Kemmer-Petiau particles with a nonminimal vector coupling: a two-fold degeneracy

T.R. Cardoso[1], L.B. Castro[2], A.S. de Castro[3]
UNESP - Campus de Guaratinguetá
Departamento de Física e Química
12516-410 Guaratinguetá SP - Brazil

[1]cardoso@feg.unesp.br
[2]benito@feg.unesp.br
[3]castro@pq.cnpq.br.

Abstract

Scalar and vector bosons in the background of one-dimensional nonminimal vector linear plus inversely linear potentials are explored in a unified way in the context of the Duffin-Kemmer-Petiau theory. The problem is mapped into a Sturm-Liouville problem with an effective singular oscillator. With boundary conditions emerging from the problem, exact bound-state solutions in the spin-0 sector are found in closed form and it is shown that the spectrum exhibits degeneracy. It is shown that, depending on the potential parameters, there may or may not exist bound-state solutions in the spin-1 sector.

Key words: Duffin-Kemmer-Petiau theory, nonminimal coupling, Klein's paradox

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1 Introduction

The first-order Duffin-Kemmer-Petiau (DKP) formalism [1]-[4] describes spin-0 and spin-1 particles and has been used to analyze relativistic interactions of spin-0 and spin-1 hadrons with nuclei as an alternative to their conventional second-order Klein-Gordon and Proca counterparts. The DKP formalism enjoys a richness of couplings not capable of being expressed in the Klein-Gordon and Proca theories [5]-[6]. Although the formalisms are equivalent in the case of minimally coupled vector interactions [7]-[9], the DKP formalism opens news horizons as far as it allows other kinds of couplings which are not possible in the Klein-Gordon and Proca theories. Nonminimal vector potentials, added by other kinds of Lorentz structures, have already been used successfully in a phenomenological context for describing the scattering of mesons by nuclei [10]-[17]. Nonminimal vector coupling with a quadratic potential [18], with a linear potential [19], and mixed space and time components with a step potential [20]-[21] and a linear potential [22]-[23] have been explored in the literature. See also Ref. [22] for a comprehensive list of references on other sorts of couplings and functional forms for the potential functions. In Refs. [22]-[23] it was shown that when the space component of the coupling is stronger than its time component the linear potential, a sort of vector DKP oscillator, can be used as a model for confining bosons.

The Schrödinger equation with a quadratic plus inversely quadratic potential, known as singular oscillator, is an exactly solvable problem [24]-[28] which works for constructing solvable models of N interacting bodies [29]-[30] as well as a basis for perturbative expansions and variational analyses for spiked harmonic oscillators [31]-[36]. Generalizations for finite-difference relativistic quantum mechanics [37] as well as for time-dependent parameters in the nonrelativistic version have also been considered [38]-[39].

The main purpose of the present article is to report on the properties of the DKP theory with nonminimal vector linear plus inversely linear potentials for spin-0 and spin-1 bosons in a unified way. The problem is mapped into an exactly solvable Sturm-Liouville problem of a Schrödinger-like equation. The effective potential resulting from the mapping has the form of the singular oscillator potential. The Schrödinger equation with quadratic plus inversely quadratic potential is indeed an exactly solvable problem. Nevertheless, only positive coefficients are involved in the well-known solution. Because we need solutions involving a repulsive as well as an attractive inverse-square term in the effective potential, the calculation including this generalization with proper boundary conditions at the origin is presented. The closed form solution for the bound states is uniquely determined with boundary conditions which emerge as a direct consequence of the equation of motion and the normalization condition, and do not have to be imposed. It is shown that the spectrum exhibits degeneracy. It is also shown that the existence of bound-state solutions for vector bosons depends on a too restrictive condition on the potential parameters.

It should be mentioned that a somewhat less general sort of problem, only with the space component of a nonminimal vector potential (erroneously called pseudoscalar potential), has already appeared in the literature [40].

2 Nonminimal vector couplings in the DKP equation

The DKP equation for a free boson is given by [4] (with units in which $\hbar = c = 1$)

$$\left(i\beta^{\mu}\partial_{\mu} - m \right) \psi = 0 \tag{1}$$

where the matrices β^μ satisfy the algebra $\beta^\mu\beta^\nu\beta^\lambda + \beta^\lambda\beta^\nu\beta^\mu = g^{\mu\nu}\beta^\lambda + g^{\lambda\nu}\beta^\mu$ and the metric tensor is $g^{\mu\nu} = \mathrm{diag}\,(1,-1,-1,-1)$. That algebra generates a set of 126 independent matrices whose irreducible representations are a trivial representation, a five-dimensional representation describing the spin-0 particles and a ten-dimensional representation associated to spin-1 particles. The second-order Klein-Gordon and Proca equations are obtained when one selects the spin-0 and spin-1 sectors of the DKP theory. A well-known conserved four-current is given by $J^\mu = \bar{\psi}\beta^\mu\psi/2$ where the adjoint spinor $\bar{\psi}$ is given by $\bar{\psi} = \psi^\dagger\eta^0$ with $\eta^0 = 2\beta^0\beta^0 - 1$. The time component of this current is not positive definite but it may be interpreted as a charge density. Then the normalization condition $\int d\tau\, J^0 = \pm 1$ can be expressed as

$$\int d\tau\, \bar{\psi}\beta^0\psi = \pm 2 \tag{2}$$

where the plus (minus) sign must be used for a positive (negative) charge.

With the introduction of nonminimal vector interactions, the DKP equation can be written as

$$\left(i\beta^\mu\partial_\mu - m - i[P,\beta^\mu]A_\mu\right)\psi = 0 \tag{3}$$

where P is a projection operator ($P^2 = P$ and $P^\dagger = P$) in such a way that $\bar{\psi}[P,\beta^\mu]\psi$ behaves like a vector under a Lorentz transformation as does $\bar{\psi}\beta^\mu\psi$. Once again $\partial_\mu J^\mu = 0$ [22]. If the potential is time-independent one can write $\psi(\vec{r},t) = \phi(\vec{r})\exp(-iEt)$, where E is the energy of the boson, in such a way that the time-independent DKP equation becomes

$$\left[\beta^0 E + i\beta^i\partial_i - \left(m + i[P,\beta^\mu]A_\mu\right)\right]\phi = 0 \tag{4}$$

In this case $J^\mu = \bar{\phi}\beta^\mu\phi/2$ does not depend on time, so that the spinor ϕ describes a stationary state.

The DKP equation is invariant under the parity operation, i.e. when $\overrightarrow{r} \to -\overrightarrow{r}$, if $\overrightarrow{A}$ changes sign, whereas A_0 remains the same. This is because the parity operator is $\mathcal{P} = \exp(i\delta_P)P_0\eta^0$, where δ_P is a constant phase and P_0 changes $\overrightarrow{r}$ into $-\overrightarrow{r}$. Because this unitary operator anticommutes with β^i and $[P,\beta^i]$, they change sign under a parity transformation, whereas β^0 and $[P,\beta^0]$, which commute with η^0, remain the same. Since $\delta_P = 0$ or $\delta_P = \pi$, the spinor components have definite parities. The charge-conjugation operation can be accomplished by the transformation $\psi \to \psi_c = \mathcal{C}\psi = CK\psi$, where K denotes the complex conjugation and C is a unitary matrix such that $C\beta^\mu = -\beta^\mu C$. Meanwhile C anticommutes with $[P,\beta^\mu]$. The matrix that satisfies these relations is $C = \exp\,(i\delta_C)\,\eta^0\eta^1$. The phase factor $\exp\,(i\delta_C)$ is equal to ± 1, thus $E \to -E$. Note also that $J^\mu \to -J^\mu$, as should be expected for a charge current and the charge-conjugation operation entails no change on A_μ. The invariance of the nonminimal vector potential under charge conjugation means that it does not couple to the charge of the boson. In other words, A_μ does not distinguish particles from antiparticles. Hence, whether one considers spin-0 or spin-1 bosons, this sort of interaction can not exhibit Klein's paradox.

For the case of spin 0, we use the representation for the β^μ matrices given by [41]

$$\beta^0 = \begin{pmatrix} \theta & \overline{0} \\ \overline{0}^T & \mathbf{0} \end{pmatrix}, \quad \beta^i = \begin{pmatrix} \widetilde{0} & \rho_i \\ -\rho_i^T & \mathbf{0} \end{pmatrix}, \quad i = 1,2,3 \tag{5}$$

where

$$
\begin{aligned}
\theta &= \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \rho_1 = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \\
\rho_2 &= \begin{pmatrix} 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \rho_3 = \begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \end{pmatrix}
\end{aligned} \tag{6}
$$

$\overline{0}$, $\widetilde{0}$ and $\mathbf{0}$ are 2×3, 2×2 and 3×3 zero matrices, respectively, while the superscript T designates matrix transposition. Here the projection operator can be written as [5] $P = \left(\beta^\mu \beta_\mu - 1\right)/3 = \mathrm{diag}\,(1,0,0,0,0)$. In this case P picks out the first component of the DKP spinor. The five-component spinor can be written as $\psi^T = (\psi_1, ..., \psi_5)$ in such a way that the time-independent DKP equation for a boson constrained to move along the X-axis, restricting ourselves to potentials depending only on x, decomposes into

$$
\begin{gathered}
\left(\frac{d^2}{dx^2} + k^2\right)\phi_1 = 0 \\
\phi_2 = \frac{1}{m}\left(E + iA_0\right)\phi_1 \\
\phi_3 = \frac{i}{m}\left(\frac{d}{dx} + A_1\right)\phi_1, \quad \phi_4 = \phi_5 = 0
\end{gathered} \tag{7}
$$

where

$$
k^2 = E^2 - m^2 + A_0^2 - A_1^2 + \frac{dA_1}{dx} \tag{8}
$$

Meanwhile,

$$
J^0 = \frac{E}{m}\left|\phi_1\right|^2, \quad J^1 = \frac{1}{m}\mathrm{Im}\left(\phi_1^* \frac{d\phi_1}{dx}\right) \tag{9}
$$

For the case of spin 1, the β^μ matrices are [42]

$$
\beta^0 = \begin{pmatrix} 0 & \overline{0} & \overline{0} & \overline{0} \\ \overline{0}^T & \mathbf{0} & \mathbf{I} & \mathbf{0} \\ \overline{0}^T & \mathbf{I} & \mathbf{0} & \mathbf{0} \\ \overline{0}^T & \mathbf{0} & \mathbf{0} & \mathbf{0} \end{pmatrix}, \quad \beta^i = \begin{pmatrix} 0 & \overline{0} & e_i & \overline{0} \\ \overline{0}^T & \mathbf{0} & \mathbf{0} & -is_i \\ -e_i^T & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \overline{0}^T & -is_i & \mathbf{0} & \mathbf{0} \end{pmatrix} \tag{10}
$$

where s_i are the 3×3 spin-1 matrices $\left(s_i\right)_{jk} = -i\varepsilon_{ijk}$, e_i are the 1×3 matrices $\left(e_i\right)_{1j} = \delta_{ij}$ and $\overline{0} = \begin{pmatrix} 0 & 0 & 0 \end{pmatrix}$, while $\mathbf{I}$ and $\mathbf{0}$ designate the 3×3 unit and zero matrices, respectively. In this representation $P = \beta^\mu \beta_\mu - 2 = \mathrm{diag}\,(1,1,1,1,0,0,0,0,0,0)$, i.e. P projects out the four upper components of the DKP spinor. With the spinor written as $\psi^T = (\psi_1, ..., \psi_{10})$, and partitioned as

$$
\begin{gathered}
\psi_8 = 0 \\
\psi_I^{(+)} = \begin{pmatrix} \psi_3 \\ \psi_4 \end{pmatrix}, \quad \psi_I^{(-)} = \psi_5 \\
\psi_{II}^{(+)} = \begin{pmatrix} \psi_6 \\ \psi_7 \end{pmatrix}, \quad \psi_{II}^{(-)} = \psi_2
\end{gathered} \tag{11}
$$

$$\psi_{III}^{(+)} = \begin{pmatrix} \psi_{10} \\ -\psi_9 \end{pmatrix}, \quad \psi_{III}^{(-)} = \psi_1$$

the one-dimensional time-independent DKP equation can be expressed as

$$\left(\frac{d^2}{dx^2} + k_\sigma^2\right) \phi_I^{(\sigma)} = 0$$

$$\phi_{II}^{(\sigma)} = \frac{1}{m}\left(E + i\sigma A_0\right) \phi_I^{(\sigma)} \tag{12}$$

$$\phi_{III}^{(\sigma)} = \frac{i}{m}\left(\frac{d}{dx} + \sigma A_1\right) \phi_I^{(\sigma)}, \quad \phi_8 = 0$$

where σ is equal to $+$ or $-$, and

$$k_\sigma^2 = E^2 - m^2 + A_0^2 - A_1^2 + \sigma \frac{dA_1}{dx} \tag{13}$$

Now the components of the four-current are

$$J^0 = \frac{E}{m} \sum_\sigma |\phi_I^{(\sigma)}|^2, \quad J^1 = \frac{1}{m} \mathrm{Im} \sum_\sigma \phi_I^{(\sigma)\dagger} \frac{d\phi_I^{(\sigma)}}{dx} \tag{14}$$

It is of interest to note that $\phi_I^{(+)}$ (in the vector sector) obeys the same equation obeyed by the first component of the DKP spinor in the scalar sector and, taking account of (9) and (14), ϕ_1 and $\phi_I^{(\sigma)}$ are square-integrable functions. Given that the interaction potentials satisfy certain conditions, we have a well-defined Sturm-Liouville problem and hence a natural and definite method for determining the possible discrete or continuous eigenvalues of the system. We also note that there is only one independent component of the DKP spinor for the spin-0 sector instead of the three required for the spin-1 sector, and that the presence of a space component of the potential might compromise the existence of solutions for spin-1 bosons when compared to the solutions for spin-0 bosons with the very same potentials (A_0 and A_1). This might happen because the solution for this class of problem consists in searching for bounded solutions for two Schrödinger equations. It should not be forgotten, though, that the equations for $\phi_I^{(+)}$ and $\phi_I^{(-)}$ are not independent because the energy of the boson, E, appears in both equations. Therefore, one has to search for bound-state solutions for $\phi_I^{(+)}$ and $\phi_I^{(-)}$ with a common energy. This amounts to say that the solutions for the spin-1 sector of the DKP theory can be obtained from a restricted class of solutions of the spin-0 sector.

3 The linear plus inversely linear potential

Now we are in a position to use the DKP equation with specific forms for vector interactions. Let us focus our attention on potentials in the linear plus inversely linear form, viz.

$$A_0 = m^2\omega_0|x| + \frac{g_0}{|x|}, \quad A_1 = m^2\omega_1 x + \frac{g_1}{x} \tag{15}$$

where the coupling constants, ω_0, ω_1, g_0 and g_1, are real dimensionless parameters. Our problem is to solve (7) and (12) for ϕ_1 (in the scalar sector) and $\phi_I^{(\sigma)}$ (in the vector sector), and to

determine the allowed energies. Although the absolute value of x in A_0 is irrelevant in the effective equations for ϕ_1 and $\phi_I^{(\sigma)}$, it is there for ensuring the covariance of the DKP theory under parity. It follows that the DKP spinor will have a definite parity and A^μ will be a genuine four-vector. In this case the first equations of (7) and (12) transmutes into

$$H_\sigma \, \Phi_\sigma = \varepsilon_\sigma \Phi_\sigma \tag{16}$$

where Φ_σ is equal to ϕ_1 for the scalar sector, and to $\phi_I^{(\sigma)}$ for the vector sector, with

$$H_\sigma = -\frac{1}{2m}\frac{d^2}{dx^2} + V_\sigma \tag{17}$$

$$\varepsilon_\sigma = \frac{E^2 - m^2 + m^2 \left[\sigma\omega_1 + 2\left(\omega_0 g_0 - \omega_1 g_1\right)\right]}{2m} \tag{18}$$

and

$$V_\sigma = \frac{1}{2} m \Omega^2 x^2 + \frac{\alpha_\sigma}{x^2} \tag{19}$$

with

$$\Omega^2 = m^2\left(\omega_1^2 - \omega_0^2\right), \quad \alpha_\sigma = \frac{g_1\left(g_1 + \sigma\right) - g_0^2}{2m} \tag{20}$$

The set (16)-(20), with $\alpha_\sigma = 0$ ($\alpha_\sigma \neq 0$) and $\Omega^2 > 0$, is precisely the Schrödinger equation for the nonrelativistic nonsingular (singular) harmonic oscillator. For $\alpha_\sigma < 0$ and $\Omega^2 = 0$ the effective potential has also a form that would make allowance for bound-state solutions with $\varepsilon_\sigma < 0$. All the remaining possibilities for α_σ and Ω^2 make the effective potential either everywhere repulsive or there appears an attractive region about $x = 0$ between potential barriers ($\alpha_\sigma < 0$ and $\Omega^2 < 0$) which lead to boson tunneling. It is well known that the singularity at $x = 0$ for $\alpha_\sigma < 0$ menaces the boson to collapse to the center [24] so that the condition $\alpha_\sigma \geq \alpha_{\mathtt{crit}} = -1/\left(8m\right)$ (with $\Omega^2 > 0$) is required for the formation of bound-state solutions. We shall limit ourselves to study the bound-state solutions.

Since V_σ is invariant under reflection through the origin ($x \to -x$), eigenfunctions with well-defined parities can be built. Thus, one can concentrate attention on the positive half-line and impose boundary conditions on Φ_σ at $x = 0$ and $x = \infty$. Continuous eigenfunctions on the whole line with well-defined parities can be constructed by taking symmetric and antisymmetric linear combinations of Φ_σ. These new eigenfunctions possess the same eigenenergy, then, in principle, there is a two-fold degeneracy. As $x \to 0$, the solution behaves as Cx^{s_σ}, where $C \neq 0$ is a constant and s_σ is a solution of the algebraic equation

$$s_\sigma(s_\sigma - 1) - 2m\alpha_\sigma = 0 \tag{21}$$

viz.

$$s_\sigma = \frac{1}{2}\left(1 \pm \sqrt{1 + 8m\alpha_\sigma}\right) \tag{22}$$

where s_σ is not necessarily a real quantity. Nevertheless, J^1 for a stationary state, as expressed by (9) and (14), is the same at all points of the X-axis and vanishes for a bound-state solution (because Φ_σ vanishes as $x \to \infty$), so we demand that $s_\sigma \in \mathbb{R}$ and so $\alpha_\sigma \geq \alpha_{\mathtt{crit}}$. Normalizability of Φ_σ also requires $s_\sigma > -1/2$ and due to the two-fold possibility of values of s_σ for $\alpha_{\mathtt{crit}} < \alpha_\sigma < 3/(8m)$, it seems that the solution of our problem can not be uniquely determined. This ambiguity can be overcome by a regularization of the potential. Following the steps of Ref.

[24], we replace $V_\sigma(x)$ by $V_\sigma(x_0)$ for $x<x_0\approx 0$ and after using the continuity conditions for Φ_σ and $d\Phi_\sigma/dx$ in the cutoff we take the limit $x_0\to 0$. It turns out that the solution with the lesser value of s_σ is suppressed relative to that one involving the greater value as $x_0\to 0$. Thus, the minus sign in (22) must be ruled out for $\alpha_\sigma\neq 0$ in such a way that $\Phi_\sigma(0)=0$. Hence, $s_\sigma=0$ or $s_\sigma\geq 1/2$. The homogeneous Dirichlet boundary condition ($\Phi_\sigma(0)=0$) is essential whenever $\alpha_\sigma\neq 0$, nevertheless it also develops for $\alpha_\sigma=0$ when $s_\sigma=1$ but not for $s_\sigma=0$. The continuity of Φ_σ at the origin excludes the possibility of an odd-parity eigenfunction for $s=0$, and effects on the even-parity eigenfunction for $s=1$ are due to the continuity of its first derivative. Effects due to the potential on $d\Phi_\sigma/dx$ in the neighbourhood of the origin can be evaluated by integrating (16) from $-\delta$ to $+\delta$ and taking the limit $\delta\to 0$. The connection formula between $d\Phi_\sigma/dx$ at the right and $d\Phi_\sigma/dx$ at the left can be summarized as

$$\lim_{\delta\to 0}\left.\frac{d\Phi_\sigma}{dx}\right|_{x=-\delta}^{x=+\delta}=2m\alpha_\sigma\lim_{\delta\longrightarrow 0}\int_{-\delta}^{+\delta}dx\,\frac{\Phi_\sigma}{x^2} \tag{23}$$

so that

$$\lim_{\delta\to 0}\left.\frac{d\Phi_\sigma}{dx}\right|_{x=-\delta}^{x=+\delta}=\begin{cases}2mC\alpha_\sigma\left.\frac{|x|^{s_\sigma-1}}{s_\sigma-1}\right|_{x=-\delta}^{x=+\delta}, & \text{for } \Phi_\sigma \text{ symmetric with } \alpha_\sigma\neq 0\\ 0, & \text{otherwise}\end{cases} \tag{24}$$

Therefore, the eigenfunctions have a first derivative continuous at the origin, except for $s_\sigma<1$ when the eigenfunction is not differentiable at the origin. Thus, the bound-state solutions for the singular oscillator potential are two-fold degenerate: for a same energy there is an odd-parity solution with a continuous first derivative, and an even-parity solution with a continuous first derivative if $s_\sigma>1$ or a discontinuous first derivative if $s_\sigma<1$. There is no odd-parity solution for $s_\sigma=0$ and there develops the homogeneous Neumann condition ($d\Phi_\sigma/dx|_{x=0}=0$), and there is no even-parity solution for $s_\sigma=1$. Thus, the bound-state solutions for the nonsingular oscillator potential are nondegenerate.

We shall now consider separately two different possibilities for Ω, namely $\Omega=0$ and $\Omega\neq 0$.

3.1 $\Omega=0$

Defining

$$z_\sigma=2\sqrt{-2m\varepsilon_\sigma}\,x \tag{25}$$

for $x>0$, and using the set (16)-(20), one obtains a special case of Whittaker´s differential equation [43]

$$\frac{d^2\Phi_\sigma}{dz_\sigma^2}+\left(-\frac{1}{4}-\frac{2m\alpha_\sigma}{z_\sigma^2}\right)\Phi_\sigma=0 \tag{26}$$

The normalizable asymptotic form of the solution as $z_\sigma\to\infty$ is $e^{-z_\sigma/2}$ with $z_\sigma>0$. Notice that this asymptotic behaviour rules out the possibility $\varepsilon_\sigma>0$, as has been pointed out already based on qualitative arguments. The exact solution can now be written as

$$\Phi_\sigma=z_\sigma^{s_\sigma}w(z_\sigma)e^{-z_\sigma/2} \tag{27}$$

where w is a regular solution of the confluent hypergeometric equation (Kummer's equation) [43]

$$z_\sigma\frac{d^2w}{dz_\sigma^2}+(b_\sigma-z_\sigma)\frac{dw}{dz_\sigma}-a_\sigma w=0 \tag{28}$$

with the definitions

$$a_\sigma = s_\sigma, \quad b_\sigma = 2s_\sigma \tag{29}$$

The general solution of (28) is expressed in terms of the confluent hypergeometric functions (Kummer's functions) ${}_1F_1(a_\sigma, b_\sigma, z_\sigma)$ (or $M(a_\sigma, b_\sigma, z_\sigma)$) and ${}_2F_0(a_\sigma, 1+a_\sigma-b_\sigma, -1/z_\sigma)$ (or $U(a_\sigma, b_\sigma, z_\sigma)$):

$$w = A_\sigma\ {}_1F_1(a_\sigma, b_\sigma, z_\sigma) + B_\sigma z_\sigma^{-a_\sigma}\ {}_2F_0(a_\sigma, 1+a_\sigma-b_\sigma, -\frac{1}{z_\sigma}), \quad b_\sigma \neq -\tilde{n}_\sigma \tag{30}$$

where $\tilde{n}_\sigma$ is a nonnegative integer. Due to the singularity of the second term at $z_\sigma = 0$, only choosing $B_\sigma = 0$ gives a behavior at the origin which can lead to square-integrable solutions. Furthermore, the requirement of finiteness for Φ_σ at $z_\sigma = \infty$ implies that the remaining confluent hypergeometric function (${}_1F_1(a_\sigma, b_\sigma, z_\sigma)$) should be a polynomial. This is because ${}_1F_1(a_\sigma, b_\sigma, z_\sigma)$ goes as e^{z_σ} as z_σ goes to infinity unless the series breaks off. This demands that $a_\sigma = -n_\sigma$, where n_σ is also a nonnegative integer. This requirement combined with (29) implies that the existence of bound-state solutions for pure inversely quadratic potentials ($|\omega_1| = |\omega_0|$) is out of question.

3.2 $\Omega \neq 0$

As for $\Omega \neq 0$, it is convenient to define the dimensionless quantity ξ,

$$\xi = m\sqrt{\Omega^2}\, x^2 \tag{31}$$

and using (16)-(20), one obtains the complete form for Whittaker´s equation [43]

$$\xi \frac{d^2\Phi_\sigma}{d\xi^2} + \frac{1}{2}\frac{d\Phi_\sigma}{d\xi} + \left(\frac{\varepsilon_\sigma}{2\sqrt{\Omega^2}} - \frac{\xi}{4} - \frac{m\alpha_\sigma}{2\xi}\right)\Phi_\sigma = 0 \tag{32}$$

The normalizable asymptotic form of the solution as $\xi \to \infty$ is given by $e^{-\xi/2}$ only for $\Omega^2 > 0$ ($|\omega_1| > |\omega_0|$). The solution for all ξ can now be written as

$$\Phi_\sigma = \xi^{s_\sigma/2} w(\xi) e^{-\xi/2} \tag{33}$$

where $w(\xi)$ is a regular solution from

$$\xi \frac{d^2 w}{d\xi^2} + (b_\sigma - \xi)\frac{dw}{d\xi} - a_\sigma w = 0 \tag{34}$$

with

$$\begin{aligned} a_\sigma &= \frac{b_\sigma}{2} - \frac{\varepsilon_\sigma}{2|\Omega|} \\ b_\sigma &= s_\sigma + 1/2 \end{aligned} \tag{35}$$

Then, w is expressed as ${}_1F_1(a_\sigma, b_\sigma, \xi)$ and in order to furnish normalizable Φ_σ, the confluent hypergeometric function must be a polynomial. This demands that $a_\sigma = -n_\sigma$ and $b_\sigma \neq -\tilde{n}_\sigma$. Note that ${}_1F_1(-n_\sigma, b_\sigma, \xi)$ is proportional to the generalized Laguerre polynomial $L_{n_\sigma}^{(b_\sigma-1)}(\xi)$, a

polynomial of degree n_σ. The requirement $a_\sigma = -n_\sigma$ combined with the top line of (35), also implies into quantized effective eigenvalues:

$$\varepsilon_\sigma = \left(2n_\sigma + s_\sigma + \frac{1}{2}\right)|\Omega|, \qquad n_\sigma = 0, 1, 2, \ldots \tag{36}$$

with

$$\Phi_\sigma \propto \xi^{s_\sigma/2} e^{-\xi/2}\, L_{n_\sigma}^{(s_\sigma - 1/2)}(\xi) \tag{37}$$

In summary, only for $\Omega^2 > 0$ ($\varepsilon_\sigma > 0$) and $\alpha_\sigma \geq \alpha_{\mathtt{crit}}$ the potentials (15) are able to furnish bound states. The spectrum is purely discrete and the solution is expressed in an exact closed form. It is instructive to note that $s_\sigma = 0$ or $s_\sigma = 1$ for the case $\alpha_\sigma = 0$ and the associated Laguerre polynomial $L_{n_\sigma}^{(-1/2)}(\xi)$ and $L_{n_\sigma}^{(+1/2)}(\xi)$ are proportional to $H_{2n_\sigma}\left(\sqrt{\xi}\right)$ or $\xi^{-1/2} H_{2n_\sigma+1}\left(\sqrt{\xi}\right)$, respectively [43]. Therefore, the solution for the effective nonsingular harmonic oscillator can be succinctly written in the customary form in terms of Hermite polynomials:

$$\varepsilon_\sigma = \left(n_\sigma + \frac{1}{2}\right)|\Omega|, \qquad n_\sigma = 0, 1, 2, \ldots \tag{38}$$

with Φ_σ proportional to

$$e^{-\xi/2} H_{n_\sigma}\left(\sqrt{\xi}\right) \tag{39}$$

The preceding analyses shows that the effective eigenvalues for the scalar sector are equally spaced with a step given by $2|\Omega|$ when the effective potential is singular at the origin. It is remarkable that the level stepping is independent of the sign and intensity of the parameter responsible for the singularity of the potential. We also note that the effective spectrum varies continuously with α_+ as far as $\alpha_+ \neq 0$. When the effective potential becomes nonsingular ($\alpha_+ = 0$) the step switches abruptly to $|\Omega|$. There is a clear phase transition when $\alpha_+ \to 0$ due the disappearance of the singularity. In the limit as $\alpha_\sigma \to 0$ the Neumann boundary condition, in addition to the Dirichlet boundary condition always present for $\alpha_+ \neq 0$, comes to the scene. This occurrence permits the appearance of even Hermite polynomials and their related eigenvalues, which intercalate among the pre-existent eigenvalues related to odd Hermite polynomials. The appearance of even Hermite polynomials makes $\Phi_+(0) \neq 0$ and this boundary condition is never permitted when the singular potential is present, even though α_+ can be small. You might also understand the lack of such a smooth transition by starting from a nonsingular potential ($\alpha_+ = 0$), when the solution of the problem involves even and odd Hermite polynomials, and then turning on the singular potential as a perturbation of the nonsingular potential. Now the “perturbative singular potential” by nature demands, if is either attractive or repulsive, that $\Phi_+(0) = 0$ so that it naturally kills the solution involving even Hermite polynomials. Furthermore, there is no degeneracy in the spectrum for $\alpha_+ = 0$.

Now we move on to match a common energy to the spin-1 boson problem. The compatibility of the solutions for $\Phi_+ = \phi_I^{(+)}$ and $\Phi_- = \phi_I^{(-)}$ demands that the quantum numbers n_+ and n_- must satisfy the relation

$$n_+ - n_- = R(\omega_0, \omega_1) \tag{40}$$

for the nonsingular potential, and

$$n_+ - n_- = \frac{1}{2}\left[R(\omega_0, \omega_1) - \sqrt{1 + 8m\alpha_+} + \sqrt{1 + 8m\alpha_-}\right] \tag{41}$$

for the singular potential. Here

$$R(\omega_0, \omega_1) = \frac{\mathrm{sgn}(\omega_1)}{\sqrt{1 - \left(\frac{\omega_0}{\omega_1}\right)^2}}$$

and $\mathrm{sgn}(\omega_1)$ stands for the sign function. For the nonsingular potential $|n_+ - n_-|$ varies from 1 to infinity (as $|\omega_0/\omega_1|$ varies from 0 to 1), whereas for the singular potential $|n_+ - n_-|$ varies from the minor to the major value between $|R(\omega_0, \omega_1)|/2$ and $\left|R(\omega_0, \omega_1) \pm \sqrt{8g_1}\right|/2$ (the $\pm$ sign corresponding to $\alpha_\pm = \alpha_{\mathtt{crit}}$). These constraints on the nodal structure of $\phi_I^{(+)}$ and $\phi_I^{(-)}$ dictate that acceptable solutions only occur for a restricted number of possibilities for the potential parameters and that no solution should be expected for $g_1 < 0$. More than this, these constraints exclude a few low-lying quantum numbers for a given set of possible solutions for the simple reason that n_+ and n_- are nonnegative integers. In conclusion, the possible bound-state solutions for spin-1 bosons exist only for a finite set of potential parameters.

To have a better understanding of the spectrum, we plot in Figure 1 the effective spectrum for the four low-lying levels as a function of α_+, for fixed ω_0 and g_0. From this figure we see the phase transition for the spin-0 spectrum at $\alpha_+ = 0$ and the severe restriction on the solutions for the spin-1 spectrum. Notice that no bound-state solution for the spin-1 spectrum exists if α_+ exceeds $|R(\omega_0, \omega_1)|/2$ and that not all the bound states have a nodeless Φ_+ for the ground state.

4 Conclusions

We approached the problem of a particle in a (3+1)-dimensional world despite the restriction to the one-dimensional movement. The motion on axis allow us to explore the physical consequences of the nonminimal vector coupling in a mathematically simpler and more physically transparent way. There is no angular momentum so that there is no spin-orbit coupling. The more general potential matrix ensuring a conserved four-current can be written in terms of well-defined Lorentz structures. For the spin-0 sector there are two scalar, two vector and two tensor terms [5], whereas for the spin-1 sector there are two scalar, two vector, a pseudoscalar, two pseudovector and eight tensor terms [6]. There is no pseudoscalar potential in the spin-0 sector of the DKP theory. In fact, the space component of a nonminimal vector potential is used in Ref. [40]. We described spin-0 particles by a five-component spinor, of which only one component is independent instead the two required in Ref. [40]. Also, the number of degrees of freedom described by the ten-component spinor in the spin-1 sector reduces correctly to three. By considering space plus time components of a nonminimal vector potential, we showed that scalar and vector bosons in the background of one-dimensional linear plus inversely linear potential is mapped into a Sturm-Liouville problem with an effective singular oscillator. The existence of solutions for spin-0 bosons requires $|\omega_1| > |\omega_0|$ and $g_1(g_1 + \sigma) - g_0^2 \geq -1/4$. In words, the time component of the nonminimal vector potential (A_0) alone can not hold bound states, and the presence of the time component in the inversely linear potential lessens the interval of parameters which provides confinement. As for spin-1 bosons, confining solutions requires $g_1 > 0$ and when compared to the solutions for spin-0 bosons with the very same potentials (A_0 and A_1) they can only be obtained from a restricted class of solutions of the spin-0 sector. It may be salutary to note that, because the differential equation (16) has a singularity at the origin for $\alpha_\sigma \neq 0$, one could expect the existence of singular solutions for Φ_σ

so that we are not free to suppose that $s_\sigma > 0$. Due to the fact that Φ_σ, whether they are even or odd, disappears at the origin for $\alpha_\sigma \neq 0$ this one-dimensional quantum-mechanical problem is degenerate and H_σ in (17) is a self-adjoint operator.

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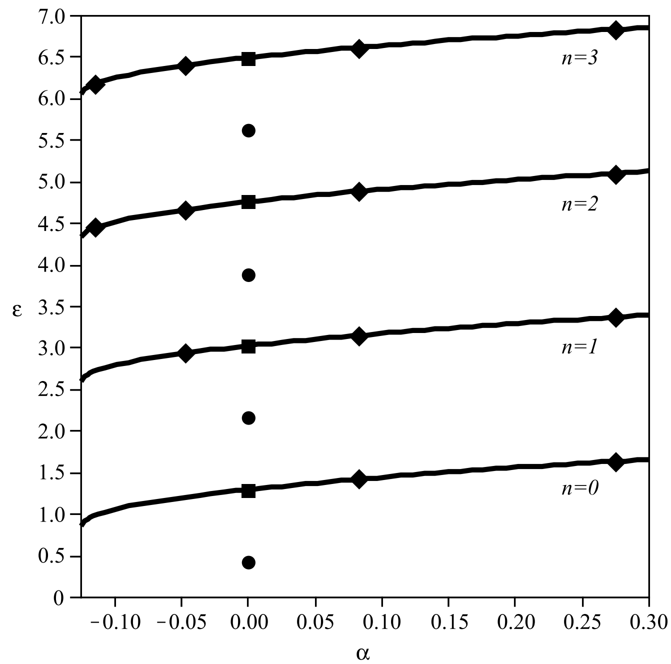

Figure 1: Effective spectrum ($\varepsilon = \varepsilon_+/m$ and $n = n_+$) for spin-0 bosons as a function of $\alpha = \alpha_+ \neq 0$ (full line) with $\omega_1 = m = 1$, $\omega_0 = 1/2$ and $g_0 = 0$. The circles and boxes stand for even and odd solutions for the case $\alpha_+ = 0$, respectively. The effective spectrum for spin-1 bosons are represented by the diamond symbols for $n_+ - n_- = +2, +1, 0, -1$ corresponding to $\alpha = -0.115$, -0.047, 0.083, 0.275 respectively.